A rough introduction

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1. Motivation

$\mathbb{Q}_p(\mu_p) \xrightarrow{\text{totall } p-1} \mathbb{Q}_p$

Let $K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge = \left(\bigcup_{n \in \mathbb{N}} \mathbb{Q}_p(\mu_{p^n})\right)^\wedge$

$\mathbb{Q}_p(\mu_{p^n}) \to \mathbb{Q}_p$ totally ram $p^{n-1}(p-1)$

$K' = \mathbb{F}_p((t^{\frac{1}{p^\infty}}))^\wedge$ → perfect in char p

Thm (Fontaine - Wintenberger):

The absolute galois gp G_K and $G_{K'}$ are isomorphic.

Rem: Why do we need/care such thing in classical p-adic Hodge theory?

One reason is that

$\mathrm{Rep}(G_{\mathbb{Q}_p})$

$\mathrm{Rep}_{\mathbb{Z}_p}(G_{K'}) \xleftrightarrow{\text{equivalence}} \phi M^{\text{ét}}_{\mathcal{O}_\varepsilon}$

$\mathcal{O}_\varepsilon = W(K')$

$\{$finitely generated $\mathcal{O}_\varepsilon$-mod M + φ-semilinear endomorphism$\}$ st. $\varphi^* M \cong M$

$\parallel$

$\mathrm{Rep}_{\mathbb{Z}_p} G_K$

The right-hand gives a concrete description of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ via Fontaine - Winten

Then extend to $\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p})$ [Recall $K = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$]

require more information ($\Gamma = \mathrm{Gal}(K/\mathbb{Q}_p)$ - action).

"Roughly - speaking"

For $g \in \Gamma$

$\varphi g = g \varphi$

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p}) \rightsquigarrow (\varphi, \Gamma)\text{-modules.}$$

Historically, $\begin{array}{c} L \\ | \\ K \end{array}$ $\rightsquigarrow$ construct an (imperfect) norm field E_L

$$\begin{array}{ccc} L & \rightsquigarrow & E_L \\ | & & \downarrow \\ K & \longrightarrow & K' \end{array}$$

$E_L^+ \subseteq \underset{\sim}{R}_L = \left\{ (x_n) \;\middle|\; \begin{array}{l} x_n \in \mathcal{O}_L \\ x_{n+1}^p = x_n \end{array} \right\}$

Later we will give this functor a new name

2. Modern view of FW Theorem

K, K' are not chosen arbitrarily.

In Scholze's language, K, K' are perfectoid fields

and $K' = K^\flat$

A1: Thm (Kedlaya-Liu, Scholze): Suppose K is a perfectoid field, $\mathrm{F\acute{E}t}(K) \cong \mathrm{F\acute{E}t}(K^b)$

$A \in \mathrm{F\acute{E}t}(K) \Leftrightarrow A = \prod_{i=1}^{n} K_i$ K_i/K is finite separable extension

3. KL's method

I: KL's pf relies heavily on their study of Banach K-algebras (K is an analytic field)

equipped with a nontrivial non-archi

ex: $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ $\mathbb{F}_p$ X

p t

Def: A Banach ring is a commutative ring with a submultiplicative norm under which it is complete.

$|xy| \leq |x| \cdot |y|$

For a Banach ring A containing a topological nilpotent unit

$x^n \to 0$ in the topology

(ex: If A is over an analytic field K)

specify A^+

power-bdd

$A^+ \subseteq A^0$ elem

A^+ is open, bounded, integrally closed

Gelfand spectrum

$$\mathcal{M}(A) \xrightarrow{i} \mathrm{Spa}(A, A^+)$$

π

$\|$ $\|$

{ ... set of multiplicative seminorms { equivalent classes of

{ the set of ... $\{ \cdot$

on A bounded by $|\cdot|$. }

ie $|\ |_r \leq C|\cdot|$.

Let $r \in \mathcal{M}(A)$

if $1 < |r(x)|$ ($x \in A^{\circ}$)

$\forall M$. $M < |r(x^n)| = |r(x)|^n \leq C|x^n| \Rightarrow |x^n| > \frac{M}{C}$

continuous valuations

$|\ | : A \to \Gamma \cup \{0\}$

s.t. $\forall a \in A^+$ $|a| \leq 1$ }

Rem: It may looks like the languge of Huber ring is more general. However, given a Tate / Analytic Huber ring, we can promote it to a Banach ring.

Rem: i). $\pi \circ i = id$. the image of i is dense

ii). π is a quotient map. $\mathcal{M}(A)$ is the maximal Hausdorff quotient of $Spa(A, A^+)$

For $v \in Spa(A, A^+)$

$\pi(v)(x) = \inf \{ |z|_{sp}^{r/s},\ r \in \mathbb{Z},\ s \in \mathbb{Z}_{>0},\ v(z^r) > v(x^s) \}$

$|z|_{sp} = \lim_{n \to \infty} |z^n|^{\frac{1}{n}}$

?

iii). In my sense. $\mathcal{M}(A)$ couldn't detect higher rk pts in $Spa(A, A^+)$.

ie. the image of i consists of rk 1 points.

iv). In Scholze's language only Spa( ) shows up. KL studies the Gel'fand spectrum

The philosophy is that:

If you want to prove something for adic spec

① prove the field case $\mathrm{Spa}(K, K^+)$ first

② prove it for $\mathcal{M}(A)$

$\mathrm{Spa}(A, A^+)$: 

r_1 is (vertical) specialization of r_0

Reduce to the case

$\mathcal{M}$ $\cdot r_0$

$\mathrm{Spa}(\mathcal{H}(v_0), \mathcal{H}^+(v_0))$
$\quad = \mathrm{Frac}(A/\mathrm{supp}(v_0))$. a field

$\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(R^\flat, R^{+\flat})$

This strategy is used for KL's proof of tilting equivalence. $\mathcal{M}(A) \cong \mathcal{M}(A^\flat)$ is more or less a direct calculation using witt vectors

II: sheafy / Nonsheafy (The motivation for uniform)

Spa (A, A^+)	Spec R
rational subsets	Spec R_f
$U\left(\frac{f_1}{g}, \dots \frac{f_n}{g}\right) = \{ x \mid x(f_i) \le x(g) \ne 0$, $(f_1, \dots f_n, g)$ genera $\}$	
$\wr$ (homeomorphic)	

Spa (B, B^+)

$$B = A\left\langle \frac{f_1}{g}, \dots \frac{f_n}{g} \right\rangle = A\langle T_1, \dots T_n\rangle \Big/ \overline{(gT_1 - f_1, \dots gT_n - f_n)}$$

The structure presheaf $\mathcal{O}$: $\Gamma(\text{Spa}(B, B^+), \mathcal{O}) = B$

We say A is **sheafy** if $\mathcal{O}$ is a sheaf.

A non-sheafy example: k analytic field.

$$A = k[T, T^{-1}, Z] \Big/ (Z^2)$$

the norm is given by (fix $0<\rho<1$)

$$\left| \sum_{\substack{n\in\mathbb{Z}\\ m=\{0,1\}}} a_{n,m} T^n Z^m \right| = \max\left\{ \max_{n\in\mathbb{Z}} \{\rho^{-n}|a_{n,0}|\}, \max_{n\in\mathbb{Z}} \{\rho^{|n|}|a_{n,1}|\} \right\}$$

$$|T^n| = \rho^{-n} \qquad |Z^n| = \rho^n$$

$R = \hat{A}$

$$\mathrm{Spa}(R\langle T\rangle, +) = U\left(\frac{T}{1}\right) \quad U\left(\frac{1}{T}\right) = \mathrm{Spa}(R\langle T^{-1}\rangle, +)$$

$f=T$, $g=1$ $\qquad$ $f=1$, $g=T$

$\longrightarrow$ Simple Laurent cover

$$\downarrow \text{cover}$$

$$\mathrm{Spa}(R, R^+)$$

Claim: $\quad R \to \underset{\mathcal{O}(U(\frac{T}{1}))\oplus \mathcal{O}(U(\frac{1}{T}))}{\overset{\shortparallel}{R\langle T\rangle \oplus R\langle T^{-1}\rangle}}$ is not inj

Idea: $R\langle T\rangle$ is the completion (separated) of $A\langle X\rangle/(X-T)$

Consider the quotient norm on $A\langle X\rangle/(X-T)$

$$|Z| = |T^n \cdot T^{-n} Z| \underset{|T|\le 1}{\le} |T^{-n}Z|_A = \rho^n$$

let $n\to\infty \Rightarrow |Z|=0$ on $A\langle T\rangle/(T-1)$.

hence is 0 in $R\langle T\rangle$ $\square$

This example is not so nice as

i) has nilpotents

ii) $(X-T)$ is not a closed ideal in $A\langle X\rangle$

Def: A Banach ring A is called <u>uniform</u>

a). $|\ |$ is power-multiplicative

b) $|\ |$ is equivalent to its spectral norm

c) $\exists\, c>0$. st $|x^2| \geq c|x|^2$

$\circledast$ d) $A^0 \subseteq A$ is bounded $\Rightarrow$ implies there are no

(d in the case A contains a uniform ~~topolog~~ nilpotent unit

$|z|_{sp}\ |z^{-1}|_{sp} = 1$

Thm (Buzzard - Verberkmoes):

A stably uniform adic Banach ring is sheafy.

Rem: Basically speakly. A uniform guarantees

$$0 \to A \to A\langle f\rangle \oplus A\langle f^{-1}\rangle \to A\langle f, f^{-1}\rangle \to 0$$

This always holds

exact

Then by some localization trick. it holds for every rational covering.

$\square$

Application: i) perfect uniform (char p) Banach-algebras are sheafy

ii) perfectoid uniform Banach-algebras are sheafy

$\mathrm{Spa}(R, R^+) \simeq \mathrm{Spa}(R^\flat, R^{\flat +})$

(reduce to case i using tilting equi)

III : Perfectoid Correspondence (finally)

Given a perfectoid field K / uniform perfectoid adic Banach-algebra (A, A^+)

i) i.e. non-arch complete field

ii) with non discrete valuation

iii) such that $\varphi: O_K/pO_K \to O_K/pO_K$ is surj).

$\mathbb{Q}_p$ is not

$\{p^n \; n \in \mathbb{Z}\}$

($A^+/p \xrightarrow{\varphi} A^+/p$ is surj.)

Def: $O_K^b = \varprojlim_{x \mapsto x^p} O_K = \varprojlim_{x \mapsto x^p} O_K/p$

$x_{i+1}^p = x_i$

$|\cdot| : (x_0, x_1, \cdots) \mapsto |(\;)^\#| = |x_0|$

$\# : O_K^b \to O_K$ is only multiplicative

Then $|O_K^b| = |O_K|$ (the valuation gp are the same

Choose $\pi \in O_K^b$ $|\pi| = |p|$

Then $K^b := O_K^b[\frac{1}{\pi}]$ and $O_K/p \cong O_{K^b}/\pi$

$= \mathrm{Frac}(O_K^b)$ indep of π

Example: $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})^b \cong \mathbb{F}_p((t^{\frac{1}{p^\infty}}))^\wedge$

$(p, p^{\frac{1}{p}}, \cdots) \longleftrightarrow t$

$\mathbb{Q}_p(\mu_{p^\infty})^{\wedge\, b} \cong \mathbb{F}_p((t^{\frac{1}{p^\infty}}))$

$(1, \mu_p, \mu_{p^2}, \dots) - 1 \longleftrightarrow t$

$\to$ The functor is not fully-faithful

Given (R, R^+) similarly define $R^\flat$, $R^{+\flat}$.

Finally we can state <u>perfectoid correspondence</u>:

- Rational localization $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(R^\flat, R^{\flat +})$

- FÉt equivalence $\mathrm{FÉt}(R) \cong \mathrm{FÉt}(R^\flat)$

Rem: "Scholze $\mathrm{Spa}(R, R^+)_{\text{ét}} \cong \mathrm{Spa}(R^\flat, R^{\flat+})_{\text{ét}}$"
$\downarrow$
étale-topology in adic space

Next goal in this subsection is to sketch

KL's proof of FÉt equivalence for fields.

This relies heavily on the use of Witt vectors

$$\begin{tikzcd} W(\mathcal{O}_K^\flat) \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \mathcal{O}_K^\flat \arrow[r] \arrow[ur, dashed, "\#"] & \mathcal{O}_K^\flat/\pi \cong \mathcal{O}_K/p \end{tikzcd}$$

θ: More precisely $[c_0] + [c_1]p + \cdots$
$\downarrow$
$c_0^\# + c_1^\# p + \cdots$

Key observation: $\ker\theta = ([z_0] + p[z_1] + \cdots)$

such that $|z_0| < 1$ $|z_1| = 1$

i.e. $z_0 \in m_{\mathcal{O}_C^\flat}$

called primitive elements of deg 1

example: $K = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$

$\pi = (p, p^{\frac{1}{p}}, \cdots) \in \mathcal{O}_C^\flat$

$\ker\theta = [\pi] - p$ $\quad \theta([\pi] - p) = \pi^\# - p = 0$

$K = \mathbb{Q}_p(\mu_{p^\infty})$

$\pi = (1, \mu_p, \mu_{p^2}, \cdots)$ $\quad \ker\theta = \sum_{i=0}^{p-1} [\pi + 1]^{i/p}$

Check: In char p case perfectoid $\Leftrightarrow$ perfect analytic field

Key input:

Thm (KL): There is an equi of categories

$\{$ perfectoid analytic fields $F\}$ $\longleftrightarrow$ $\{(L, I) \mid L$ is perfect analytic of char p. $I \subseteq W(\mathcal{O}_L)$ a principal ideal generated by primitive degree 1 elements$\}$

$F \rightsquigarrow (F^\flat, \ker\theta)$

$\mathrm{Frac}(W(\mathcal{O}_L)/I) \longleftarrow (L, I)$

(over $\mathbb{Q}_p$)

Rem: Similar results for perfectoid uniform Banach adic-algebras

Recently, similar results show up in Bhatt and Scholze's prismatic coho theory:

$$\text{Perfect prisms} \overset{\text{equivalent}}{\longleftrightarrow} \text{perfectoid rings}$$

$$(A, I) \longmapsto A/I$$

$$(\underbrace{W(\mathcal{O}_K^\flat)}, I) \qquad\qquad C_K$$

Proof of A1: char 0   char p

Step I:

$$\begin{tikzcd}
E \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, leftrightarrow] & (M, IW(\mathcal{O}_M)) \\
F \arrow[r, leftrightarrow] & (L, I) \arrow[u, "\text{finite extension}"']
\end{tikzcd} \qquad [E:F] = [M:L]$$

Step II: Now fix the completion of an algebraic closure M of L

Thm (Kedlaya): E is alge closed.

If $K^\flat$ is alge closed

$\Downarrow$

K is alge closed

$$\begin{tikzcd}
E \arrow[r, leftrightarrow] & (M, IW(\mathcal{O}_M)) \arrow[d, dash] \\
K_i \arrow[u] \arrow[r, leftrightarrow] & H_i \arrow[d, dash, "\text{finite}"] \\
F \arrow[u, leftrightarrow] \arrow[r, leftrightarrow] & (L, I)
\end{tikzcd}$$

$\mathrm{Gal}_F \cong \mathrm{Gal}_L$

$E = \overline{\bigcup K_i}$ we need to show $\bigcup K_i$ is separably closed

" If $\alpha \in E$ separable over F Use Krasner's lemma

$\Rightarrow$ ~~∃~~ β $\beta \in \bigcup K_i$ $F[\alpha] = F[\beta]$ "   □

4. Asides : Schelze's pf of AI :

$$\begin{tikzcd}
K_{f\acute{e}t} \arrow[r, "\sim", "\text{"Almost purity"}"'] \arrow[d, "\wr"', "A \longmapsto A^{\circ a}"] & K^{\circ}_{a f\acute{e}t} \arrow[r, "\sim", "\text{vanishing}"'] & (K^{\circ}/\pi)_{f\acute{e}t} \arrow[d, "\wr", "\text{of cotangent complex}"'] \\
K^{b}_{f\acute{e}t} \arrow[r, "\sim"] & K^{b,\circ}_{a.f\acute{e}t} \arrow[r, "\sim"] & (K^{b,\circ}/\pi^{b})_{f\acute{e}t}
\end{tikzcd}$$

(K can be replaced by Perf_K : uniform perfectoid Banach K-alg.

Rem : Almost purity relies highly on "deeply ramified". that's why we require the valuation gp is non-discrete in the def

5. Various (relative) Robba rings.

Based on KL's theorem. Suppose $C^\flat$ is alge closed field of char p. Then (alge closed) perfectoid field K such that

K is called untilting of $C^\flat$

$K^\flat \cong C^\flat$ is parametrized by primitive elements of deg 1.

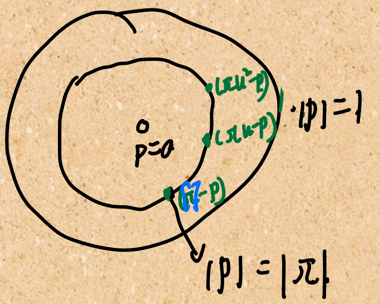
$\Downarrow$

give such parametrization a geometric structure

Idea: view p as a variable

$$A_{\inf} = W(\mathcal{O}_{C^\flat})$$

Y:

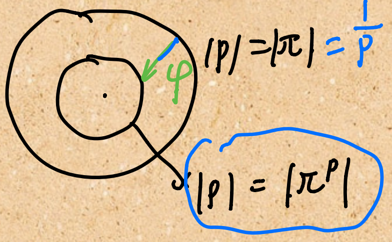


Recall $\pi = p^\flat$

$$\varphi([\pi]) = [\pi^p]$$

$|\pi| = p^{-1}$

$\varphi(p) = p$

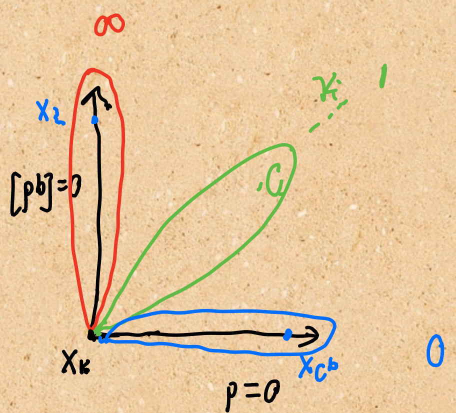


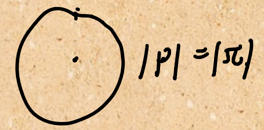
- When parametrize untilts, we actually want to module Frobenius

$$(K,\ i: C^\flat \cong K^\flat) \big/ (\varphi(K), \varphi\circ i)$$

$\therefore$ want $\tilde{Y} = Y/\varphi^{\mathbb{Z}}$

The picture can also be drawn in another way:

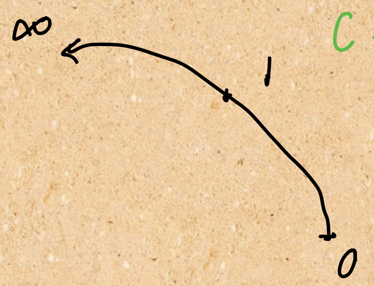




"$x_K : p=0$
$[\pi]=0$"

$$Y = \mathrm{Spa}\, W(\mathcal{O}_{C^\flat}) \setminus \{x_K\}$$

$C : [\pi] - p = 0$



$$\kappa : Y \longrightarrow [0, \infty]$$

$$x \longmapsto \frac{\log |[\pi](x_{gen})|}{\log |p(x_{gen})|}$$

$$X = Y_{(0,\infty)}/\varphi^{\mathbb{Z}} \longrightarrow \text{(adic) Fargues-Fontaine curve}$$

$$H^0(Y_{(0,\infty)}, \mathcal{O}) = \varprojlim_{\substack{[a,b],\ a\neq 0\\ b<\infty}} H^0(Y_{[a,b]}, \mathcal{O})$$

$$y \in Y_{[a,b]} : \quad x(p)^a \leq x([\pi]) \leq x(p)^b$$

$$\therefore H^0(Y_{[a,b]}, \mathcal{O}) = A_{\inf}\left\langle \frac{p}{[\pi]^{1/b}}, \frac{[\pi]^{\frac{1}{a}}}{p} \right\rangle \left[\frac{1}{p}\right]$$

$$H^0(Y_{(0,\infty)}, \mathcal{O}) = \varprojlim_{\substack{a>0\\ b<\infty}} A_{\inf}\left\langle \frac{p}{[\pi]^{1/b}}, \frac{[\pi]^{\frac{1}{a}}}{p} \right\rangle \left[\frac{1}{p}\right]$$

This ring is independent of π we choose.

It is called B in Fargues-Fontaine's theory

$$H^0(\mathcal{X}, \mathcal{O}) = B^{\varphi = Id} = \mathbb{Q}_p$$

$$X_{FF} := \underset{d \geq 0}{\mathrm{Proj}} \left(B^{\varphi = p^d} \right) \longrightarrow \text{Schematic FF-curve}$$

Thm (FF): • X_{FF} is a regular noetherian scheme.

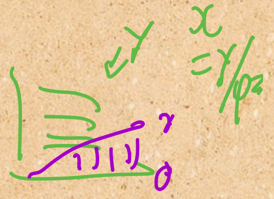
of Krull dimension 1.

3 methods • F-F $\Leftarrow$ • Vector bundles $\mathcal{M} = \bigoplus \mathcal{O}\left(\frac{d_i}{n_i}\right)$ (HN filtration)

$\frac{d_1}{n_1} > \cdots > \frac{d_k}{n_k}$

• K-L

• Scho-Farg



Relation with Kedlaya's Robba ring:

$$\tilde{R}^{int, r} = H^0\left(Y_{[0,r]}, \mathcal{O}\right)$$

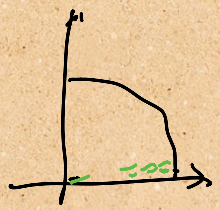
$$= \left\{ \sum_{n=0}^{\infty} p^n [c_n] \;\middle|\; c_n \in C^b,\ c_n \pi^{n/r} \to 0 \right\}$$

$$\tilde{R}^{bd, r} = \tilde{R}^{int, r}\left[\frac{1}{p}\right] = \left\{ \sum_{n >> 0} p^n [c_n] \;\middle|\; c_n \in C^b,\ c_n \pi^{n/r} \to 0 \text{ when } n \to \infty \right\}$$

$$\tilde{R}^{r} = \widehat{\tilde{R}^{bd,r}} \qquad (\text{Frechét-completion})$$

$$\tilde{R} = \varinjlim_{r\to 0} \tilde{R}^{r}$$

$X = Y_{(0,\infty)}/\varphi^{\mathbb{Z}}$



Prop: $\{$vector bundles on $X_{FF}\}$

$$\updownarrow$$

$\{\varphi$-modules on $Y_{(0,\infty)}\}$

$$\updownarrow$$

$\{\varphi$-modules over $\tilde{R}\}$

Thm: (Kedlaya)

$$\{\lambda_1 \geq \cdots \lambda_n \mid n\in\mathbb{N}, \lambda_i\in\mathbb{Q}\} \xleftarrow{\ \sim\ } \varphi\text{-modules over } \tilde{R}/\sim$$

$$(\lambda_1, \cdots \lambda_n) \longmapsto \bigoplus_{i=1}^{n} \tilde{R}(-\lambda_i)$$

This gives another proof of FF-theorem via some effort (equivalence in prop preserves semi-stability)

□